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ASTRONOMY and ASTROPHYSICS

MY NEW PROBLEMS Volume II

EDITURA DIDACTICĂ ȘI PEDAGOGICĂ S.A.

PROPOSALS

Problem 51. Spaceship to the Alpha Centauri Star

Space Shuttle leaves the Earth in a straight line and uniform motion towards the Centauri Star α, with the velocity $\vec{u}$ relative to the Earth as shown in the drawing in Figure 1. The Centauri Star α is considered to be at rest relative to the Earth. Immediately after the launch of the Space Shuttle, from it, a Cosmic Rocket takes off towards the Centauri Star α.

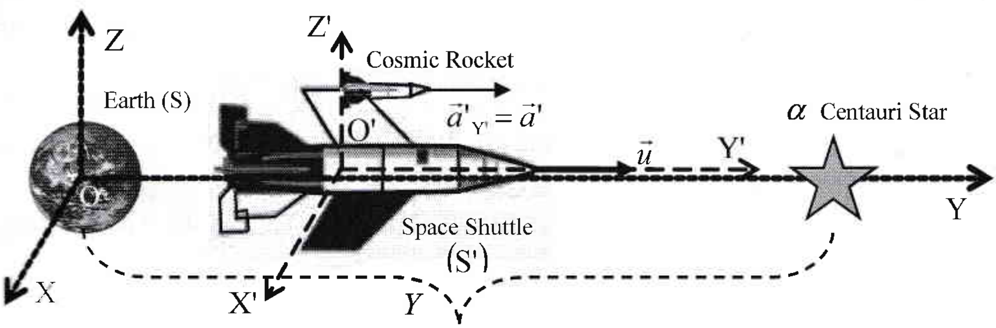

Fig. 1

The propulsion force of the Space Rocket provides it, towards the Centauri Star α, a constant relative acceleration, $\vec{a}'_{Y'} = \vec{a}'$, considered in relation to the inertial system attached to the Space Shuttle.

At the time of launch of the Space Shuttle, when the origins O and, respectively O', of the two inertial reference systems, S and respectively S', associated with the Earth and respectively the Space Shuttle, coincide, the clocks in the two reference systems have been synchronized, so that $t = t' = 0$.

The Space Rocket takes off from the Space Shuttle from the point corresponding to the origin of the inertial system associated with the Space Shuttle. The speed of light in a vacuum is known, c.

a) Establish the relationship between the acceleration of the Space Rocket in relation to the Earth, $a_Y = a$, and the speed of the Space Rocket in relation to the Earth, $v_Y = v$.

b) Establish the speed law of the Cosmic Rocket, in relation to the Earth, $v_Y(t) = v(t)$.

c) Establish the law of acceleration of the Cosmic Rocket, in relation to the Earth, $a_Y(t) = a(t)$.

d) Establish the law of the position coordinate of the Cosmic Rocket, in relation to the Earth, $y(t)$. *It is known that*:

$$\int \frac{A+B\cdot x}{C+D\cdot x}\cdot \mathrm{d}x = \frac{1}{D}\left(B\cdot x + \frac{\Delta}{D}\cdot \ln\left|C+D\cdot x\right|\right);\ \Delta = AD-BC.$$

e) *Determine* he distance between the Earth and the Centauri Star α , Y, if the duration of the Cosmic Rocket's travel between the Earth and the Centauri Star, relative to the observer on Earth, is T.

f) *Determine* the speed of the Cosmic Rocket, in relation to the Earth, at the time of its arrival at the Centauri Star α .

g) *Determine* the duration of the movement of the Cosmic Rocket between the Earth and the Centauri Star α , timed by the observer O' on the Space Shuttle, T', as well as the distance traveled by the Cosmic Rocket, in relation to the observer O' on the Space Shuttle, from the moment the Rocket took off from the Space Shuttle and until upon arrival at Centauri Star α , Y'.

h) *Particularize* the results from points **b, c** and **d**, for the case $u << c$, knowing that:

$$\ln(1+x) \approx x - \frac{1}{2}x^2,\ \text{dacă}\ x << 1.$$

Solving

a) According to the notations in figure 2, corresponding to a certain moment in the evolution of the Cosmic Rocket, related to the two inertial reference systems, it follows:

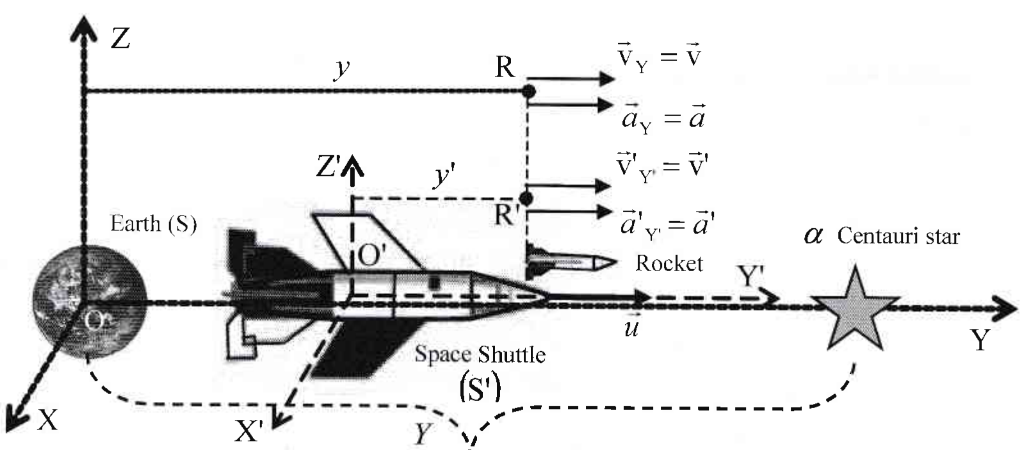

Fig. 2

$$\mathrm{R}'(0, y', 0; t');\ \mathrm{R}(0, y, 0; t);$$

$$y=\frac{y'+ut'}{\sqrt{1-\frac{u^2}{c^2}}}>y';\ t=\frac{t'+\frac{uy'}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}>t';$$

$$\mathrm{v}_{Y}=\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\frac{\mathrm{d}y'+u\cdot\mathrm{d}t'}{\sqrt{1-\frac{u^2}{c^2}}}}{\frac{\mathrm{d}t'+\frac{u}{c^2}\mathrm{d}y'}{\sqrt{1-\frac{u^2}{c^2}}}}=\frac{\mathrm{d}y'+u\cdot\mathrm{d}t'}{\mathrm{d}t'+\frac{u}{c^2}\mathrm{d}y'}=\frac{\left(\frac{\mathrm{d}y'}{\mathrm{d}t'}+u\right)\cdot\mathrm{d}t'}{\left(1+\frac{u}{c^2}\cdot\frac{\mathrm{d}y'}{\mathrm{d}t'}\right)\cdot\mathrm{d}t'};$$

$$\mathrm{v}_{Y}=\frac{\left(\frac{\mathrm{d}y'}{\mathrm{d}t'}+u\right)}{\left(1+\frac{u}{c^2}\cdot\frac{\mathrm{d}y'}{\mathrm{d}t'}\right)};\ \mathrm{v}_{Y}=\frac{\mathrm{v}'_{Y'}+u}{\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)};$$

$$\mathrm{dv}_{Y}=\frac{\mathrm{d}\left(\mathrm{v}'_{Y'}+u\right)\cdot\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)-\left(\mathrm{v}'_{Y'}+u\right)\cdot\mathrm{d}\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)}{\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)^2};$$

$$\mathrm{dv}_{Y}=\frac{\mathrm{dv}'_{Y'}\cdot\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)-\left(\mathrm{v}'_{Y'}+u\right)\cdot\frac{u}{c^2}\mathrm{dv}'_{Y'}}{\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)^2};$$

$$\mathrm{dv}_{Y}=\frac{1-\frac{u^2}{c^2}}{\left(1+\frac{u}{c^2}\cdot\mathrm{v}'_{Y'}\right)^2}\cdot\mathrm{dv}'_{Y'};$$

$$t=\frac{t'+\frac{uy'}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}};$$

$$\mathrm{d}t = \frac{\mathrm{d}t' + \frac{u}{c^2}\cdot \mathrm{d}y'}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1+\frac{u}{c^2}\cdot\frac{\mathrm{d}y'}{\mathrm{d}t'}}{\sqrt{1-\frac{u^2}{c^2}}}\cdot \mathrm{d}t';$$

$$\mathrm{d}t = \frac{1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}}{\sqrt{1-\frac{u^2}{c^2}}}\cdot \mathrm{d}t';$$

$$a_Y = \frac{\mathrm{dv}_Y}{\mathrm{d}t};\; a'_{Y'} = \frac{\mathrm{dv}'_{Y'}}{\mathrm{d}t'};$$

$$a_Y = \frac{\dfrac{1-\frac{u^2}{c^2}}{\left(1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}\right)^2}\cdot \mathrm{dv}'_{Y'}}{\dfrac{1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}}{\sqrt{1-\frac{u^2}{c^2}}}\cdot \mathrm{d}t'} = \frac{\dfrac{1-\frac{u^2}{c^2}}{\left(1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}\right)^2}}{\dfrac{1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}}{\sqrt{1-\frac{u^2}{c^2}}}}\cdot\frac{\mathrm{dv}'_{Y'}}{\mathrm{d}t'};$$

$$a_Y = a'_{Y'}\cdot\frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\left(1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}\right)^3};\; \mathrm{v}_Y = \frac{\mathrm{v}'_{Y'}+u}{\left(1+\frac{u}{c^2}\cdot \mathrm{v}'_{Y'}\right)};$$

$$\mathrm{v}'_{Y'} = \frac{\mathrm{v}_Y - u}{\left(1-\frac{u}{c^2}\cdot \mathrm{v}_Y\right)};$$

$$a_Y = a'_{Y'}\cdot\frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\left(1+\frac{u}{c^2}\cdot\dfrac{\mathrm{v}_Y-u}{1-\frac{u}{c^2}\cdot \mathrm{v}_Y}\right)^3};$$

$$a_{Y} = a'_{Y'} \cdot \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u}{c^2} \cdot \frac{v_{Y} - u}{c^2 - u\,v_{Y}} \cdot c^2\right)^3};\; a_{Y} = a'_{Y'} \cdot \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u\,v_{Y} - u^2}{c^2 - u\,v_{Y}}\right)^3};$$

$$a_{Y} = a'_{Y'} \cdot \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(\frac{c^2 - u\,v_{Y} + u\,v_{Y} - u^2}{c^2 - u\,v_{Y}}\right)^3};$$

$$a_{Y} = a'_{Y'} \cdot \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(\frac{c^2 - u^2}{c^2 - u\,v_{Y}}\right)^3};\; a_{Y} = a'_{Y'} \cdot \frac{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(\frac{1 - \frac{u^2}{c^2}}{1 - \frac{u\,v_{Y}}{c^2}}\right)^3};$$

$$a_{Y} = a'_{Y'} \cdot \frac{\left(1 - \frac{u\,v_{Y}}{c^2}\right)^3}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}};$$

$$a'_{Y'} = a';\; a_{Y} = a;\; v_{Y} = v;\; a = a' \cdot \frac{\left(1 - \frac{u\,v}{c^2}\right)^3}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}.$$

b)

$$v_{Y} = \frac{u + v'_{Y'}}{1 + \frac{u}{c^2} \cdot v'_{Y'}};$$

$$v'_{Y'} = a'_{Y'} \cdot t';$$

$$v_{Y} = \frac{u + a'_{Y'}\,t'}{1 + \frac{u}{c^2} \cdot a'_{Y'}\,t'};$$

$$t=\frac{t'+\frac{uy'}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}>t';\quad t'=\frac{t-\frac{uy}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}<t;\ y=ut;$$

$$t'=\frac{t-\frac{u}{c^2}ut}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{t\left(1-\frac{u^2}{c^2}\right)}{\sqrt{1-\frac{u^2}{c^2}}};$$

$$t'=t\cdot\sqrt{1-\frac{u^2}{c^2}};$$

$$v_Y=\frac{u+a'_{Y'}t'}{1+\frac{u}{c^2}\cdot a'_{Y'}t'};$$

$$v_Y=\frac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}=v_Y(t);\quad a'_{Y'}=a';\ v_Y=v;$$

$$v_Y=\frac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}=v_Y(t).$$

c) From the previous equations:

$$a_Y=a'_{Y'}\cdot\frac{\left(1-\frac{u\,v_Y}{c^2}\right)^3}{\left(1-\frac{u^2}{c^2}\right)^{3/2}};\quad a_Y=a'_{Y'}\cdot\frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\left(1+\frac{u}{c^2}\cdot\frac{v_Y-u}{1-\frac{u}{c^2}\cdot v_Y}\right)^3},$$

$$v_Y=\frac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}=v_Y(t);$$

results:

$$\frac{\mathrm{v}_{Y}-u}{1-\frac{u}{c^{2}}\cdot \mathrm{v}_{Y}}=\frac{\dfrac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}{1+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}-u}{1-\dfrac{u}{c^{2}}\cdot\dfrac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}{1+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}};$$

$$\frac{\mathrm{v}_{Y}-u}{1-\frac{u}{c^{2}}\cdot \mathrm{v}_{Y}}=\frac{\dfrac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t-u-\frac{u^{2}}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}{1+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}}{\dfrac{1+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t-\frac{u}{c^{2}}\left(u+a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t\right)}{1+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}};$$

$$\frac{\mathrm{v}_{Y}-u}{1-\frac{u}{c^{2}}\cdot \mathrm{v}_{Y}}=\frac{\dfrac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t-u-\frac{u^{2}}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t}{1}}{\dfrac{1+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t-\frac{u}{c^{2}}\left(u+a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t\right)}{1}};$$

$$\frac{\mathrm{v}_{Y}-u}{1-\frac{u}{c^{2}}\cdot \mathrm{v}_{Y}}=\frac{\dfrac{a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t\cdot\left(1-\frac{u^{2}}{c^{2}}\right)}{1}}{\dfrac{1-\frac{u^{2}}{c^{2}}+\frac{u}{c^{2}}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t-\frac{u}{c^{2}}\left(a'_{Y'}\cdot\sqrt{1-\frac{u^{2}}{c^{2}}}\cdot t\right)}{1}};$$

$$\frac{\mathrm{v}_{\mathrm{Y}}-u}{1-\frac{u}{c^2}\cdot \mathrm{v}_{\mathrm{Y}}}=\frac{\frac{a'_{\mathrm{Y'}}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t\cdot\left(1-\frac{u^2}{c^2}\right)}{1}}{\frac{1-\frac{u^2}{c^2}}{1}};$$

$$\frac{\mathrm{v}_{\mathrm{Y}}-u}{1-\frac{u}{c^2}\cdot \mathrm{v}_{\mathrm{Y}}}=\frac{\frac{a'_{\mathrm{Y'}}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1}}{\frac{1}{1}};$$

$$\frac{\mathrm{v}_{\mathrm{Y}}-u}{1-\frac{u}{c^2}\cdot \mathrm{v}_{\mathrm{Y}}}=a'_{\mathrm{Y'}}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t;$$

$$a_{\mathrm{Y}}=a'_{\mathrm{Y'}}\cdot\frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\left(1+\frac{u}{c^2}\cdot\frac{\mathrm{v}_{\mathrm{Y}}-u}{1-\frac{u}{c^2}\cdot \mathrm{v}_{\mathrm{Y}}}\right)^3};$$

$$a_{\mathrm{Y}}=a'_{\mathrm{Y'}}\cdot\frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\left(1+\frac{u}{c^2}\cdot a'_{\mathrm{Y'}}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t\right)^3}=a_{\mathrm{Y}}(t).$$

d) Knowing that:

$$\mathrm{v}_{\mathrm{Y}}=\frac{u+a'_{\mathrm{Y'}}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1+\frac{u}{c^2}\cdot a'_{\mathrm{Y'}}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}=\mathrm{v}_{\mathrm{Y}}(t);$$

$$\mathrm{v}_{\mathrm{Y}}=\frac{\mathrm{d}\,y}{\mathrm{d}\,t},$$

results:

$$y=\int \mathrm{v}_{Y}\mathrm{d}t=\int\frac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}\cdot\mathrm{d}t;$$

$$\int\frac{A+B\cdot x}{C+D\cdot x}\cdot\mathrm{d}x=\frac{1}{D}\left(B\cdot x+\frac{\Delta}{D}\cdot\ln\left|C+D\cdot x\right|\right);$$

$$\Delta=AD-BC;$$

$$A=u;\ B=a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}};\ C=1;\ D=\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}};$$

$$x=t;$$

$$\Delta=\frac{u^2}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}-a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}=-a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot\left(1-\frac{u^2}{c^2}\right);$$

$$\Delta=-a'_{Y'}\cdot\left(1-\frac{u^2}{c^2}\right)^{3/2};$$

$$y(t)=\frac{1}{\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}}\left(a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t-\frac{a'_{Y'}\cdot\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}}\cdot\ln\left|1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t\right|\right);$$

$$y(t)=\frac{c^2}{u}\cdot\frac{1}{a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}}\left(a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t-\frac{c^2}{u}\cdot\frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\ln\left|1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t\right|\right);$$

$$y(t)=\frac{c^2}{u\cdot a'_{Y'}}\left(a'_{Y'}\cdot t-\frac{c^2\cdot\sqrt{1-\frac{u^2}{c^2}}}{u}\cdot\ln\left|1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t\right|\right);$$

$$y(t)=\frac{c^2}{u}\left(t-\frac{c^2\cdot\sqrt{1-\frac{u^2}{c^2}}}{u\cdot a'_{Y'}}\cdot\ln\left|1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t\right|\right).$$

e)

$$t=T;\ y(t)=y(T)=Y;$$

$$y(t)=\frac{c^2}{u\cdot a'_{Y'}}\left(a'_{Y'}\cdot t-\frac{c^2\cdot\sqrt{1-\frac{u^2}{c^2}}}{u}\cdot\ln\left|1+\frac{u}{c^2}\cdot a'_{Y'}\cdot t\cdot\sqrt{1-\frac{u^2}{c^2}}\right|\right).$$

$$Y=\frac{c^2}{u\cdot a'_{Y'}}\left(a'_{Y'}\cdot T-\frac{c^2\cdot\sqrt{1-\frac{u^2}{c^2}}}{u}\cdot\ln\left|1+\frac{u}{c^2}\cdot a'_{Y'}\cdot T\cdot\sqrt{1-\frac{u^2}{c^2}}\right|\right).$$

f)

$$t=T;\ \mathrm{v}_Y(t)=\mathrm{v}_Y(T)=\mathrm{v}_{Y,\max};$$

$$\mathrm{v}_Y=\frac{u+a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}{1+\frac{u}{c^2}\cdot a'_{Y'}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot t}=\mathrm{v}_Y(t);$$

$$\mathrm{v}_{Y,\max}=\frac{u+a'_{Y'}\cdot T\cdot\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{u}{c^2}\cdot a'_{Y'}\cdot T\cdot\sqrt{1-\frac{u^2}{c^2}}}.$$

g)

$$Y'=\frac{a'_{Y'}T'^2}{2};\ Y=\frac{Y'+uT'}{1+\frac{u\,\mathrm{v}'_{Y',\max}}{c^2}};\ \mathrm{v}'_{Y',\max}=a'_{Y'}T';$$

$$Y=\frac{Y'+uT'}{1+\frac{ua'_{Y'}T'}{c^2}};\ Y=\frac{\frac{a'_{Y'}T'^2}{2}+uT'}{1+\frac{ua'_{Y'}T'}{c^2}};$$